

# Conformal properties of four-gluon planar amplitudes and Wilson loops

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## Abstract

We present further evidence for a dual conformal symmetry in the four-gluon planar scattering amplitude in  $\mathcal{N} = 4$  SYM. We show that all the momentum integrals appearing in the perturbative on-shell calculations up to four loops are dual to true conformal integrals, well defined off shell. Assuming that the complete off-shell amplitude has this dual conformal symmetry and using the basic properties of factorization of infrared divergences, we derive the special form of the finite remainder previously found at weak coupling and recently reproduced at strong coupling by AdS/CFT. We show that the same finite term appears in a weak coupling calculation of a Wilson loop whose contour consists of four light-like segments associated with the gluon momenta. We also demonstrate that, due to the special form of the finite remainder, the asymptotic Regge limit of the four-gluon amplitude coincides with the exact expression evaluated for arbitrary values of the Mandelstam variables.

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# 1 Introduction

Gluon and quark scattering amplitudes have long been the subject of numerous studies in QCD. These amplitudes have a very non-trivial structure consisting of infrared singular and finite parts. In physical infrared safe observables like inclusive cross-sections the former cancel in the sum of all diagrams [1] while the latter produce a rather complicated function of the kinematic variables. The infrared singular part of the scattering amplitude has a universal structure [2, 3, 4] which is intimately related to the properties of Wilson loops [5]. This leads to an evolution equation for the amplitude as a function of the infrared cutoff [6, 7, 8] which is governed, in the planar limit, by the so-called cusp anomalous dimension of the Wilson loop [9, 10]. This anomalous dimension first emerged in the studies of *ultraviolet* cusp singularities of Wilson loops [11] (see also [12] and references therein) and its relation to *infrared* asymptotics in gauge theories was discovered in [5, 13, 14]. The cusp anomalous dimension is very important in QCD since it controls the asymptotic behavior of various gauge invariant quantities like the double-log (Sudakov) asymptotics of form factors, the logarithmic scaling of the anomalous dimension of higher-spin operators, the gluon Regge trajectory, etc. However, unlike the singular part, the finite part of the gluon scattering amplitude in QCD is much more involved, being given in terms of certain special functions of the Mandelstam variables.

Recently, a lot of attention has been paid to the problem of calculating gluon scattering amplitudes in the context of the maximally supersymmetric Yang-Mills theory ( $\mathcal{N} = 4$  SYM). These amplitudes have been extensively studied in perturbation theory where they have been constructed using state-of-the-art unitarity cut techniques [15, 16, 17, 18, 19]. The results of these studies concern both the divergent and finite parts of the amplitude. Although the main subject of the present paper is the finite part, we start with a brief review of the IR singularities.

## 1.1 Infrared divergences in gluon scattering amplitudes

Unlike a generic gauge theory,  $\mathcal{N} = 4$  SYM is ultraviolet finite. Despite this UV finiteness, the gluon scattering amplitudes are still IR divergent, even though the singular structure is much simpler compared to QCD, due to the fact that the coupling does not run. As in QCD, the dependence on the IR cutoff is determined by the cusp anomalous dimension.

The notion of cusp anomalous dimension was initially introduced [11, 12] in the context of a Wilson loop evaluated over a closed (Euclidean) contour with a cusp (see Fig. 1). By definition,  $\Gamma_{\text{cusp}}(a, \vartheta)$  is a function of the coupling constant  $a$  and the cusp angle  $\vartheta$  describing the dependence of the Wilson loop on the *ultraviolet* cutoff. Later on it was realized [5, 13] that the same quantity  $\Gamma_{\text{cusp}}(a, \vartheta)$  determines the *infrared* asymptotics of scattering amplitudes in gauge theories, for which a dual Wilson loop is introduced with an integration contour  $C$  uniquely defined by the particle momenta. The cusp angle  $\vartheta$  is related to the scattering angles and it takes large values in Minkowski space,  $|\vartheta| \gg 1$ . In this limit,  $\Gamma_{\text{cusp}}(a, \vartheta)$  scales linearly in  $\vartheta$  to all loops [14]

$$\Gamma_{\text{cusp}}(a, \vartheta) = \vartheta \Gamma_{\text{cusp}}(a) + O(\vartheta^0), \quad (1)$$

where  $\Gamma_{\text{cusp}}(a)$  is a function of the coupling constant only. In what follows we shall use the term *cusp anomalous dimension* in this restricted sense, to denote the quantity  $\Gamma_{\text{cusp}}(a)$ . In a dimensionally regularized four-gluon scattering amplitude, the IR poles exponentiate and  $\Gamma_{\text{cusp}}(a)$  controls the coefficient of the double pole in the exponent.

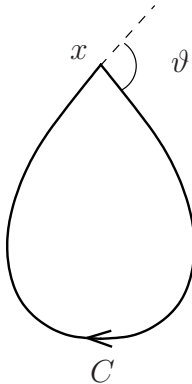


Figure 1: An example of a Wilson loop with a cusp at the point  $x$ .

The two-loop expression for  $\Gamma_{\text{cusp}}(a, \vartheta)$  and  $\Gamma_{\text{cusp}}(a)$  in a generic (supersymmetric) Yang-Mills theory can be found in [14, 20]. Among the important results of the recent studies in  $\mathcal{N} = 4$  SYM was the calculation of  $\Gamma_{\text{cusp}}(a)$  at three [17] and at four [18, 21] loops. The three-loop value of  $\Gamma_{\text{cusp}}(a)$  confirmed the maximal transcendentality conjecture of [22] based on advanced calculations of twist-two anomalous dimensions in QCD [23]. The four-loop value provided support for a conjecture [24] about the form of the all-loop cusp anomalous dimension derived from Bethe Ansatz equations.

## 1.2 Finite part of the four-gluon amplitude

As mentioned before, the finite part of scattering amplitudes in QCD is a very complicated object. The situation turns out to be radically simpler in  $\mathcal{N} = 4$  SYM. One of the main results of Bern *et al* is a very interesting all-loop iteration conjecture about the IR finite part of the color-ordered planar amplitude which takes the surprisingly simple form

$$\ln \mathcal{M}_4 = [\text{IR divergences}] + \frac{\Gamma_{\text{cusp}}(a)}{4} \ln^2 \frac{s}{t} + \text{const} , \quad (2)$$

where  $a = g^2 N / (8\pi^2)$  is the coupling constant and  $s$  and  $t$  are the Mandelstam kinematic variables.<sup>1</sup> Compared to QCD, we see the following important simplifications: (i) the complicated functions of  $s/t$  appearing in QCD expressions are replaced by the elementary function  $\ln^2(s/t)$ ; (ii) no higher powers of logs appear in  $\ln \mathcal{M}_4$  at higher loops; (iii) the coefficient of  $\ln^2(s/t)$  is determined by  $\Gamma_{\text{cusp}}(a)$ , just like the coefficient of the double log in the IR divergent part. This conjecture has been verified up to three loops for four-gluon amplitudes in [17] (a similar conjecture for  $n$ -gluon amplitudes [17] has been confirmed for  $n = 5$  at two loops in [25, 26]).

Recently Alday and Maldacena [27] have proposed a prescription for studying gluon scattering amplitudes at strong coupling via AdS/CFT. Their calculation produced exactly the same form (2) of the finite part, with the strong-coupling value of  $\Gamma_{\text{cusp}}(a)$  obtained from the semiclassical analysis of [28]. This result constitutes a non-trivial test of the AdS/CFT correspondence. Nevertheless, on the gauge theory side the deep reason for the drastic simplification of the finite part remains unclear.

In this paper we present arguments that the specific form of the finite part of the four-gluon amplitude, both in perturbation theory and at strong coupling, may be related to a hidden

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<sup>1</sup>To avoid the appearance of an imaginary part in  $\ln \mathcal{M}_4$ , it is convenient to choose  $s$  and  $t$  negative.

conformal symmetry of the amplitude. To avoid misunderstandings, we wish to stress that this is not a manifestation (at least, not in an obvious way) of the underlying conformal symmetry of the  $\mathcal{N} = 4$  SYM theory.

### 1.3 Dual conformal symmetry

The starting point of our discussion is the on-shell planar four-gluon scattering amplitude given in terms of IR divergent momentum scalar-like integrals, regularized dimensionally by going to  $D = 4 - 2\epsilon_{\text{IR}}$  ( $\epsilon_{\text{IR}} < 0$ ) dimensions. These integrals possess a surprising symmetry which we shall refer to as *dual conformal symmetry*. Its presence was first revealed, up to three loops, in [29] and was later on confirmed at four loops in [18].<sup>2</sup> Also, it was recently used in [19] as a guiding principle to construct the five-loop amplitude.

The central observation of [29] is that in order to uncover this dual conformal symmetry of the on-shell integrals, one has to go through three steps: (i) assign ‘off-shellness’ (or ‘virtuality’) to the external momenta entering the integrand; (ii) set  $\epsilon_{\text{IR}} = 0$ ; (iii) make the change of variables

$$p_i = x_i - x_{i+1} . \quad (3)$$

In this way the integrals cease to diverge because the virtualities of the external momenta serve as an IR cutoff. The resulting four-dimensional integrals become manifestly conformal in the dual space description with ‘coordinates’  $x_i$ . Here we stress that these are not the original coordinates in position space (the Fourier counterparts of the momenta), but represent the momenta themselves. The conformal symmetry is then easily seen by doing conformal inversion  $x^\mu \rightarrow x^\mu/x^2$  and counting the conformal weights at the integration points.

The authors of [18] and [19] have noticed that some of the dual conformal four- and five-loop integrals that they could list, in reality do not contribute to the amplitude. In the present paper we give the explanation of this fact, namely, all the non-contributing integrals are in fact divergent even off shell.

It is important to realize that taking off shell the integrals which appear in the on-shell calculations of Bern *et al*, in order to reveal their dual conformal properties, does not mean that we know the exact form of the off-shell amplitude regularized by the virtuality of the external legs. Indeed, there are indications that the complete off-shell amplitude may involve more integrals which vanish in the on-shell regime (see the discussion in Section 6). Nevertheless, inspired by the strong evidence for a conformal structure from the perturbative calculations of Bern *et al*, we make the conjecture that the full off-shell amplitude is conformal. Then, combining this assumption with the basic properties of factorization of four-gluon amplitudes into form factors and the exponentiation of the latter (valid also in the off-shell regime), we deduce that the special form of the finite remainder discussed above is a direct consequence of the dual conformal invariance. This is one of the main points of the present paper.

### 1.4 Light-like Wilson loops

Another point we would like to make concerns the recent proposal of Alday and Maldacena for a string dual to the four-gluon amplitudes [27]. They identify  $\ln \mathcal{M}_4$  with the area of the world-sheet of a classical string in AdS space, whose boundary conditions are determined by the gluon

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<sup>2</sup>The four-loop amplitude was constructed in [18] without any assumption about dual conformal invariance. However, it turned out that only dual conformal integrals appear in it.

momenta. Quite interestingly, they use exactly the same change of variables (3) which they interpret as a T-duality transformation on the string world-sheet. Then they apply conformal transformations in the dual space to relate different solutions to the string equations of motion. Remarkably, their calculation looks very similar to that of the expectation value of a Wilson loop made out of four light-like segments  $(x_i, x_{i+1})$  in  $\mathcal{N} = 4$  SYM at strong coupling [30, 31].

Motivated by these findings, in the present paper we revisit the calculation of such a light-like Wilson loop at weak coupling for generic values of  $s$  and  $t$ . We establish the correspondence between the IR singularities of the four-gluon amplitude and the UV singularities of the Wilson loop and extract the finite part of the latter at one loop. Remarkably, our result is again of the form (2). This indicates that the duality between gluon amplitudes and Wilson loops discussed by Alday and Maldacena is also valid at weak coupling.

The relationship between gluon scattering amplitudes and light-like Wilson loops has already been investigated at weak coupling in QCD in the context of the Regge limit  $s \gg -t > 0$  [32]. Here we examine the asymptotic behavior of the  $\mathcal{N} = 4$  SYM four-gluon amplitude in the Regge regime and demonstrate that, due to the special form of the finite remainder, the amplitude is Regge exact. This means that the contribution of the gluon Regge trajectory to the amplitude coincides with its exact expression evaluated for arbitrary values of  $s$  and  $t$ . This property is in sharp contrast with QCD, where the simplicity of the Regge limit is lost if the amplitude is considered in a general regime.

## 2 Perturbative structure of planar four-gluon scattering amplitudes. Evidence for off-shell conformal symmetry

In this section we discuss some properties of the loop integrals appearing in the perturbative calculation of the planar four-gluon scattering amplitude. It is expressed in terms of dimensionally regularized Feynman integrals with the external legs on shell. However, if one takes the legs off shell and restricts the integrals to four dimensions, they exhibit an unexpected dual conformal symmetry.

The planar four-gluon amplitude in  $\mathcal{N} = 4$  SYM at one loop [33, 34] and two loops [15, 16] is expressed in terms of ladder (or scalar box) integrals. Such integrals have long been known to have special properties. In particular, when treated off shell in four dimensions, they are conformally covariant [35]. At three loops [17] the amplitude is given by three-loop ladder integrals as well as one new type of integral, the so-called ‘tennis court’ (see Fig. 2). In [29] it was shown that the ‘tennis court’ integral is also conformally covariant.

As an example, consider the one-loop scalar box integral

$$I^{(1)} = \int \frac{d^D k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} . \quad (4)$$

It is a function of four gluon momenta  $p_i$  such that  $\sum_{i=1}^4 p_i = 0$ . When the external legs are put on shell,  $p_i^2 = 0$ , the integral becomes infrared divergent and needs to be regularized. One way to do this is to change the dimension from  $D = 4$  to  $D = 4 - 2\epsilon_{\text{IR}}$ ,  $\epsilon_{\text{IR}} < 0$  (dimensional regularization). Another way is to have the external legs slightly off shell,  $p_i^2 \neq 0$  and later on to take the limit  $p_i^2 \rightarrow 0$ . In the latter approach we can keep  $D = 4$  which allows us to reveal the

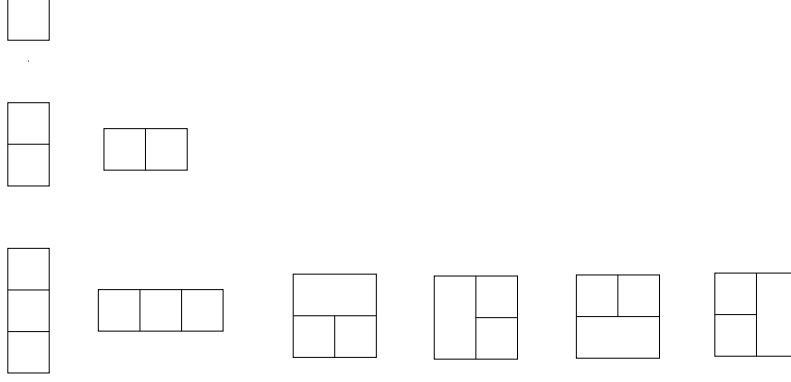


Figure 2: Integral topologies up to three loops

conformal properties of the integral. This is done by introducing a dual ‘coordinate’ description (see Fig. 3).

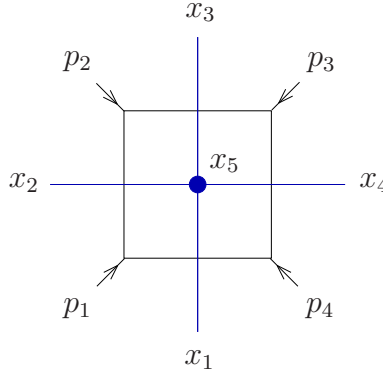


Figure 3: Dual diagram for the one-loop box

We define dual variables  $x_i$  by (with  $x_{jk} \equiv x_j - x_k$ )

$$p_1 = x_{12}, \quad p_2 = x_{23}, \quad p_3 = x_{34}, \quad p_4 = x_{41}, \quad k = x_{15}, \quad (5)$$

so that  $\sum_i p_i = 0$ . We stress that these are not the coordinates in the original position space (the Fourier counterparts of the momenta), but simply a reparametrization of the momenta (note the ‘wrong’ dimension of mass of  $x_{i,i+1}$ ). In terms of these new variables the integral (4) takes the form

$$I^{(1)} = \int \frac{d^D x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}. \quad (6)$$

It is manifestly invariant under translations and rotations of the  $x$  coordinates. It is also covariant under conformal inversion,

$$x^\mu \longrightarrow \frac{x^\mu}{x^2} : \quad \frac{1}{x_{ij}^2} \longrightarrow \frac{x_i^2 x_j^2}{x_{ij}^2}, \quad d^D x \longrightarrow \frac{d^D x}{(x^2)^D}, \quad (7)$$

provided that the transformation of the propagators at the integration point  $x_5$  is exactly compensated by the transformation of the measure. This can only happen if  $D = 4$ . Then the

integral is equal to a conformally covariant factor multiplied by a function of the conformally invariant cross-ratios  $u$  and  $v$ ,<sup>3</sup>

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (8)$$

Thus we have

$$I^{(1)} = \frac{i\pi^2}{x_{13}^2 x_{24}^2} \Phi^{(1)}(u, v), \quad (9)$$

where the function  $\Phi^{(1)}$  is expressed in terms of logs and dilogs [36].

We wish to point out that this conformal invariance of the loop integrals is *a priori* unrelated (or at least not related in an obvious way) to the conformal symmetry of the underlying  $\mathcal{N} = 4$  SYM theory. For this reason we prefer to call this property of the loop integrals ‘dual conformal invariance’.

The conformal properties of the higher-order scalar box integrals and of the ‘tennis court’ can be seen in the same way. For the ‘tennis court’ there is the new feature that the integrand contains a numerator (denoted by a dashed line). It is a positive power of  $x_{35}^2$  connecting the external point 3 to the integration point 5 where more than four propagators join together (Fig. 4). The rôle of this numerator is to compensate the surplus of conformal weight due to the propagators at this internal point, thus maintaining conformal covariance.

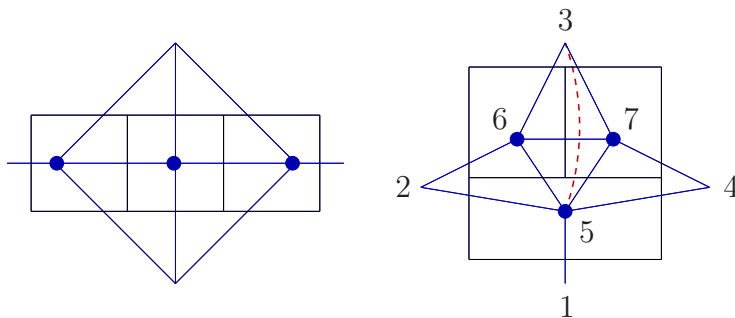


Figure 4: Dual diagrams for the three-loop box and for the ‘tennis court’ with its numerator

At four loops the conformal pattern continues. The amplitude was constructed in [18] and it is again expressed entirely in terms of dual conformal integrals. The authors of [18] give a list of ten such planar integrals which are non-vanishing on shell and have only log (no power) singularities. All of them satisfy the formal requirement of dual conformal covariance, namely that the conformal weight of the integrand at each integration point should be four. However, it turns out that of the ten integrals, eight contribute to the amplitude and two do not. We are now able to give a simple explanation: By inspection one can see that the two non-contributing integrals are in fact divergent in four dimensions (even off shell). Thus they do not have well-defined conformal properties, since they require a regulator to exist at all. Furthermore, all the integrals which do contribute to the amplitude are finite and hence conformal in four dimensions.

We illustrate the nature of the divergence in Fig. 5. The integrals in this figure correspond to the integrals (d) and (d’) of [18]. They differ only in the distribution of the numerator factors. The first integral is finite off shell in four dimensions and contributes to the amplitude. The second integral contains the four-loop structure indicated at the bottom of the diagram. When all four

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<sup>3</sup>We denote the cross-ratios by  $u$  and  $v$  and reserve  $s$  and  $t$  to denote the Mandelstam variables.



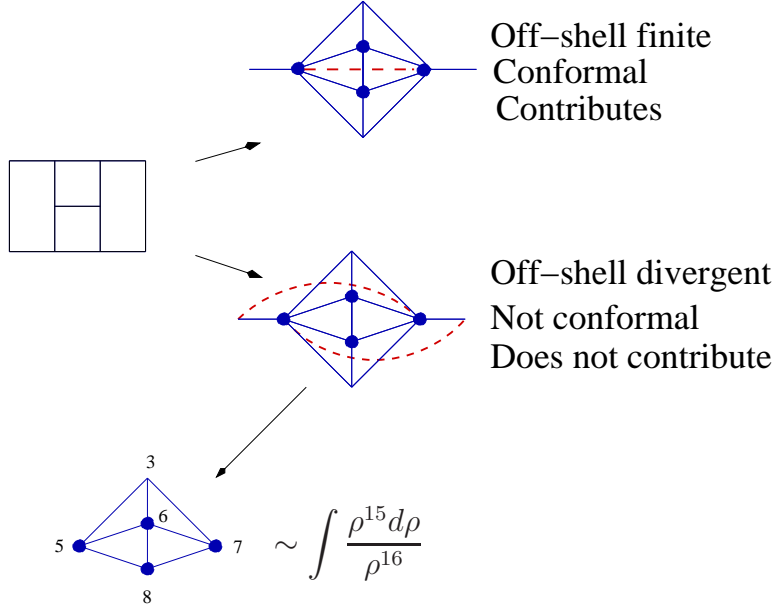


Figure 5: Examples of off-shell finite and divergent integrals with the same topology but different numerators

integration points  $x_{5,6,7,8}$  approach an external point, e.g.  $x_3$ , connected by three propagators, the integral scales at short distances  $\rho^2 = x_{53}^2 + x_{63}^2 + x_{73}^2 + x_{83}^2 \rightarrow 0$  as<sup>4</sup>

$$\int \frac{d^4 x_5 d^4 x_6 d^4 x_7 d^4 x_8}{x_{53}^2 x_{63}^2 x_{73}^2 x_{83}^2 x_{56}^2 x_{67}^2 x_{78}^2 x_{58}^2 x_{68}^2} \sim \int \frac{\rho^{15} d\rho}{\rho^{16}} \quad (10)$$

and therefore it is divergent as  $\rho \rightarrow 0$ . The first integral does not suffer from this divergence due to the different numerator structure which softens the behaviour in the equivalent region of integration by one power of  $\rho^2$ .

In principle, the fact that some integrals are divergent off shell in four dimensions is not a problem in the study of dimensionally regularized on-shell amplitudes. It is, however, striking that an integral contributes to the amplitude if and only if it has well-defined conformal properties off shell. The pattern also continues to five loops. In [19] the five-loop amplitude was constructed based on the assumption that it be a linear combination of dual conformal integrals. The authors found 59 such integrals of which only 34 contribute to the amplitude. Once again, all 34 contributing integrals are truly conformal, i.e. they are finite off shell in four dimensions, and the 25 non-contributing integrals are divergent. The divergences can be either of the same form as above (i.e. a four-loop subdivergence) or of the type where all five integration points approach an external point.

Another remarkable property of the amplitude up to five loops is that the dual conformal integrals all come with coefficient  $\pm 1$ . It seems reasonable to conjecture that these properties hold to all loops, i.e. that the all-order planar four-gluon amplitude has the form (after dividing by the tree amplitude)<sup>5</sup>,

$$\mathcal{M}_4 = 1 + \sum_{\mathcal{I}} \sigma(\mathcal{I}) a^{l(\mathcal{I})} \mathcal{I}, \quad (11)$$

<sup>4</sup>Here it is tacitly assumed that for  $s, t$  negative and with the external legs off-shell, the Feynman integrals can be analytically continued to the Euclidean region.

<sup>5</sup>Here we are using the same conventions as [17].



where the sum runs over all true dual conformal integrals,  $\mathcal{I}$ . In this formula,  $l(\mathcal{I})$  is the loop order of the integral,  $a$  is the coupling and  $\sigma(\mathcal{I}) = \pm 1$ . The sign can be determined graph by graph by using the unitarity cut method as described in [19] but a simple rule for it is still lacking.

Thus we have seen that there is very strong evidence for a dual conformal structure behind the planar four-gluon amplitude to all orders. Here we wish to stress once more that knowing the on-shell amplitudes as given in terms of dimensionally regularized integrals and simply changing the regulator does not mean that we have obtained the full off-shell amplitude. The latter may involve further integrals. At the present stage we can only conjecture that they are also conformal up to terms which vanish with the removal of the infrared cutoff.

### 3 Factorization and exponentiation of infrared singularities. Finite part and dual conformal invariance

In this section we analyze the general structure of infrared divergences of four-gluon scattering amplitudes in the on-shell and off-shell regimes. In both cases, they factorize in the planar limit into a product of form factors in the  $s$ - and  $t$ -channels. The latter are known to have a simple exponential form governed by the cusp anomalous dimension and two other (subleading) anomalous dimensions [37, 38]. This completely determines the infrared divergent part of the four-gluon amplitude, but leaves the finite part arbitrary. We show that the simple requirement of off-shell dual conformal invariance fixes the finite part to exactly the form (2) observed at weak [17] and strong [27] coupling.

#### 3.1 One-loop example

Let us first consider the one-loop four-gluon amplitude (divided by the tree amplitude) given by the one-loop box integral  $I^{(1)}$  defined in (4):

$$\mathcal{M}_4 = 1 + aM^{(1)} + O(a^2) = 1 - \frac{a}{2}stI^{(1)} + O(a^2) , \quad (12)$$

where the coupling is given by

$$a = \frac{g^2 N}{8\pi^2} \quad (13)$$

and  $s = (p_1 + p_2)^2$  and  $t = (p_2 + p_3)^2$  are the Mandelstam variables. In the on-shell regime ( $p_i^2 = 0$ ) we can use the dimensional regularization scheme where the integral is multiplied by a normalization factor including the regularization scale  $\mu$ . Expanding the integral in powers of the regulator  $\epsilon_{\text{IR}}$  one finds [17]

$$M_{\text{on-shell}}^{(1)} = \left( \frac{\mu_{\text{IR}}^2}{-s} \right)^{\epsilon_{\text{IR}}} \left[ -\frac{2}{\epsilon_{\text{IR}}^2} - \frac{1}{\epsilon_{\text{IR}}} \ln \frac{s}{t} + 4\zeta_2 + O(\epsilon_{\text{IR}}) \right] , \quad (14)$$

where  $\mu_{\text{IR}}^2$  is related to the dimensional regularization scale as

$$\mu_{\text{IR}}^2 = 4\pi e^{-\gamma} \mu^2 \quad (15)$$

and  $\gamma$  is the Euler constant. This amplitude can be split into a divergent and a finite part in such a way that the latter does not depend on the scale  $\mu_{\text{IR}}$ :

$$M_{\text{on-shell}}^{(1)} = D_{\text{on-shell}}^{(1)} + F_{\text{on-shell}}^{(1)} + O(\epsilon_{\text{IR}}) , \quad (16)$$

with the divergent part given by

$$D_{\text{on-shell}}^{(1)} = -\frac{1}{\epsilon_{\text{IR}}^2} \left[ \left( \frac{\mu_{\text{IR}}^2}{-s} \right)^{\epsilon_{\text{IR}}} + \left( \frac{\mu_{\text{IR}}^2}{-t} \right)^{\epsilon_{\text{IR}}} \right] , \quad (17)$$

and the finite part given by

$$F_{\text{on-shell}}^{(1)} = \frac{1}{2} \ln^2 \frac{s}{t} + 4\zeta_2 . \quad (18)$$

As mentioned in the introduction, the dependence of the on-shell scattering amplitude  $\mathcal{M}_4^{\text{on-shell}}$  on the IR cutoff  $\mu_{\text{IR}}^2$  is governed by an evolution equation. In  $\mathcal{N} = 4$  SYM and in the planar limit, this equation takes the simple form [37]

$$\left( \frac{\partial}{\partial \ln \mu_{\text{IR}}^2} \right)^2 \ln \mathcal{M}_4^{\text{on-shell}} = -\Gamma_{\text{cusp}}(a) + O(\epsilon_{\text{IR}}) , \quad (19)$$

where  $\Gamma_{\text{cusp}}(a)$  is the cusp anomalous dimension of a Wilson loop, Eq. (1). Thus, substituting  $\mathcal{M}_4^{\text{on-shell}} = 1 + aM_{\text{on-shell}}^{(1)}$  into (19) we obtain  $\Gamma_{\text{cusp}}(a) = 2a + O(a^2)$ .<sup>6</sup>

Let us now redo the same calculation, but keeping the integral (4) in four dimension and using instead a small ‘virtuality’ of the external legs  $p_i^2 = -m^2$  as an infrared cutoff. The off-shell amplitude has been calculated in [34] and is again given by the one-loop box integral as in (12). This time we can use the conformal expression (9) of the integral with the function  $\Phi^{(1)}(u, v)$  from [36]. According to (5) and (8), the conformal cross-ratios  $u$  and  $v$  are now given by<sup>7</sup>

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \frac{p_1^2 p_3^2}{st} = \frac{m^4}{st} , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = \frac{p_4^2 p_2^2}{st} = \frac{m^4}{st} . \quad (20)$$

It is then not hard to find the expansion of the integral in terms of the cutoff  $m$ :

$$M_{\text{off-shell}}^{(1)} = -\frac{1}{2} \ln^2 \left( \frac{m^4}{st} \right) - \zeta_2 + O(m) . \quad (21)$$

It can again be split into a divergent and a finite part as follows:

$$M_{\text{off-shell}}^{(1)} = D_{\text{off-shell}}^{(1)} + F_{\text{off-shell}}^{(1)} + O(m) , \quad (22)$$

with

$$D_{\text{off-shell}}^{(1)} = -\ln^2 \left( \frac{m^2}{-s} \right) - \ln^2 \left( \frac{m^2}{-t} \right) , \quad (23)$$

and

$$F_{\text{off-shell}}^{(1)} = \frac{1}{2} \ln^2 \frac{s}{t} - \zeta_2 . \quad (24)$$

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<sup>6</sup>Note that the so-called soft anomalous dimension  $\gamma_K$  [4, 39] is related to  $\Gamma_{\text{cusp}}$  as follows:  $\gamma_K = 2\Gamma_{\text{cusp}}$ .

<sup>7</sup>We are not obliged to set all virtualities  $p_i^2$  equal and we do it here only for simplicity. Keeping  $p_i^2$  independent would allow us to examine the full off-shell conformal structure.

Notice that the double-pole singularity of the dimensionally regularized amplitude (17) has been replaced by a double-log (log-squared) singularity in the cutoff  $m$ . As before, the finite part (24) does not depend on the IR cutoff. We remark that the finite part is the same in both schemes, except for the scheme-dependent additive constant [40].

In the off-shell regime, the evolution equation for the scattering amplitude  $\mathcal{M}_4^{\text{off-shell}} = 1 + aM_{\text{off-shell}}^{(1)} + O(a^2)$  takes the form [38]

$$\left(\frac{\partial}{\partial \ln m^2}\right)^2 \ln \mathcal{M}_4^{\text{off-shell}} = -2\Gamma_{\text{cusp}}(a) + O(m), \quad (25)$$

giving again  $\Gamma_{\text{cusp}} = 2a + O(a^2)$ . Notice the characteristic difference of a factor of 2 in (25) compared to the on-shell expression (19). Its origin can be understood as follows [38]. The amplitude  $M_{\text{off-shell}}^{(1)}$  is given by the one-loop scalar box integral in which the loop momentum  $k^\mu$  is integrated over the whole phase space. The divergent contribution to  $M_{\text{off-shell}}^{(1)}$ , Eq. (23), comes from two regions in the phase space: the so-called soft region,  $k^\mu = O(m)$ , and the infrared (or ‘ultra-soft’ [41]) region,  $k^\mu = O(m^2/Q)$ , with the hard scale  $Q \sim \sqrt{|s|}, \sqrt{|t|}$ . Each region produces a double-logarithmic contribution  $\sim (\ln m^2)^2$  which translates into  $(-\Gamma_{\text{cusp}}(a))$  in the right-hand side of (25). This explains the factor 2 in (25). In the on-shell regime, one finds that the infrared region does not exist while the soft region provides the same  $(-\Gamma_{\text{cusp}}(a))$  contribution to the right-hand side of the evolution equation (19).

## 3.2 Generalization to all orders

In this subsection we give a review of some generic properties of gluon amplitudes in gauge theory and some special properties of the four-gluon amplitudes in  $\mathcal{N} = 4$  SYM. The first property we will need in the following discussion is the factorization of the infrared singular amplitude, both on and off shell, into Sudakov form factors  $M_{gg \rightarrow 1}$  for a colorless boson decaying into two gluons. Further, the singularities of the latter are known to exponentiate [42, 43], thus determining the structure of the infrared singularities of the four-gluon amplitude up to a few constants. However, this leaves the finite part of the four-gluon amplitude arbitrary. At the end of this subsection we show that the additional assumption of dual conformal symmetry in the off-shell regime fixes the finite part up to an additive constant.

### 3.2.1 On-shell regime

In the on-shell regime ( $p_i^2 = 0$ ,  $D = 4 - 2\epsilon_{\text{IR}}$ ) and in the planar limit the  $n$ -gluon amplitude takes the factorized form

$$\mathcal{M}_n = \mathcal{F}_n \prod_{i=1}^n \left[ M_{gg \rightarrow 1} \left( \frac{\mu_{\text{IR}}^2}{-s_{i,i+1}}, a, \epsilon_{\text{IR}} \right) \right]^{1/2}. \quad (26)$$

Here the factor  $\mathcal{F}_n$  is finite and the kinematic variables are  $s_{i,i+1} = (p_i + p_{i+1})^2$ .

Restricting (26) to the case of interest  $n = 4$ , we can write the factorized amplitude in the following form:

$$\mathcal{M}_4^{\text{on-shell}} = \mathcal{F}_4^{\text{on-shell}} M_{gg \rightarrow 1}^{\text{on-shell}} \left( \frac{\mu_{\text{IR}}^2}{-s}, \epsilon_{\text{IR}} \right) M_{gg \rightarrow 1}^{\text{on-shell}} \left( \frac{\mu_{\text{IR}}^2}{-t}, \epsilon_{\text{IR}} \right). \quad (27)$$

The form factor  $M_{gg \rightarrow 1}^{\text{on-shell}}$  satisfies an evolution equation [37]. In a finite theory, where the coupling does not run, the solution of this evolution equation has a particularly simple exponential form:

$$\ln M_{gg \rightarrow 1}^{\text{on-shell}} \left( \frac{\mu_{\text{IR}}^2}{-s} \right) = -\frac{1}{2} \sum_{l=1}^{\infty} a^l \left( \frac{\mu_{\text{IR}}^2}{-s} \right)^{l\epsilon_{\text{IR}}} \left[ \frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon_{\text{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\text{IR}}} + A^{(l)} \right] + O(\epsilon_{\text{IR}}) . \quad (28)$$

Here  $\Gamma_{\text{cusp}}^{(l)}$  are the coefficients in the perturbative expansion of the cusp anomalous dimension  $\Gamma_{\text{cusp}}(a)$ . The constants  $G^{(l)}$  and  $A^{(l)}$  are regularization scheme dependent.

Replacing  $M_{gg \rightarrow 1}^{\text{on-shell}}$  in (27) by its expression (28), we find the following splitting of the log of the four-gluon amplitude into a divergent and a finite parts:

$$\ln \mathcal{M}_4^{\text{on-shell}} = \ln M_{gg \rightarrow 1}^{\text{on-shell}} \left( \frac{\mu_{\text{IR}}^2}{-s}, \epsilon_{\text{IR}} \right) + \ln M_{gg \rightarrow 1}^{\text{on-shell}} \left( \frac{\mu_{\text{IR}}^2}{-t}, \epsilon_{\text{IR}} \right) + \ln \mathcal{F}_4^{\text{on-shell}} \left( \frac{s}{t} \right) + O(\epsilon_{\text{IR}}) . \quad (29)$$

This relation generalizes the one-loop result (16) to higher loops. A unique feature of this particular splitting of  $\ln \mathcal{M}_4^{\text{on-shell}}$  is that the  $\mu_{\text{IR}}^2$  dependence is restricted to the divergent part of the amplitude (coming from the form factors). From (29) one can extract the cusp anomalous dimension applying (19). At the same time, the finite part, being independent of the IR scale  $\mu_{\text{IR}}^2$ , is a function of the remaining dimensionless variable  $s/t$  only.

One of the central results of Ref. [17] is the conjecture that in the case of  $\mathcal{N} = 4$  SYM the finite part of the four-gluon amplitude, as defined by the splitting (29), has a very simple all-order form:

$$\ln \mathcal{F}_4^{\text{on-shell}} = \frac{\Gamma_{\text{cusp}}(a)}{4} \ln^2 \frac{s}{t} + \text{const} . \quad (30)$$

One may say that the one-loop finite part (18) exponentiates to all orders. This conjecture has been verified in [17] up to three loops. The same form of the finite part has been found in [27] at strong coupling. In what follows we give an argument in favor of this conjecture, based on the assumption of conformal invariance of the amplitude in the off-shell regime.

### 3.2.2 Off-shell regime

In the off-shell regime ( $p_i^2 = -m^2$ ,  $D = 4$ ) the four-gluon amplitude still factorizes as indicated in (27), but now the Sudakov form factor [42, 44] has the following infrared divergent structure [38]:

$$\ln M_{gg \rightarrow 1}^{\text{off-shell}} \left( \frac{m^2}{-s} \right) = -\frac{1}{2} \Gamma_{\text{cusp}}(a) \ln^2 \left( \frac{m^2}{-s} \right) + G^{\text{off-shell}}(a) \ln \left( \frac{m^2}{-s} \right) + \text{const} + O(m) . \quad (31)$$

The main difference from the on-shell regime is that  $\Gamma_{\text{cusp}}$  appears with a factor of 2 off shell (compare the double-log derivatives of (28) and (31) with respect to the IR scale). The reason for this is the same as for the one-loop off-shell amplitude (25) – the Sudakov form factor receives an additional contribution from the infrared region  $k^\mu = O(m^2/(\sqrt{-s}))$ .

The splitting of the four-gluon amplitude takes the form (cf. (29))

$$\ln \mathcal{M}_4^{\text{off-shell}} = \ln M_{gg \rightarrow 1}^{\text{off-shell}} \left( \frac{m^2}{-s} \right) + \ln M_{gg \rightarrow 1}^{\text{off-shell}} \left( \frac{m^2}{-t} \right) + \ln \mathcal{F}_4^{\text{off-shell}} \left( \frac{s}{t} \right) + O(m) . \quad (32)$$

This relation generalizes the one-loop result (22) to higher loops. Substituting (31) into (32), we obtain

$$\begin{aligned} \ln \mathcal{M}_4^{\text{off-shell}} &= -\frac{1}{4}\Gamma_{\text{cusp}}(a) \ln^2 \frac{m^4}{st} + G^{\text{off-shell}}(a) \ln \frac{m^4}{st} \\ &\quad -\frac{1}{4}\Gamma_{\text{cusp}}(a) \ln^2 \frac{s}{t} + \ln \mathcal{F}_4^{\text{off-shell}}\left(\frac{s}{t}\right) + \text{const} + O(m) . \end{aligned} \quad (33)$$

### 3.2.3 Off-shell conformal symmetry and the form of the finite part

As we have seen earlier, there is substantial evidence from perturbation theory that the planar four-gluon amplitude in the on-shell regime possesses a hidden dual conformal structure. We may take this as an indication that the full off-shell amplitude discussed above will exhibit the same symmetry. As discussed in subsection 1.3 this is not obvious because the full off-shell amplitude may contain additional integrals which vanish in the on-shell regime. Nevertheless, let us adopt this conformal symmetry as an assumption. Then, up to terms which vanish as  $m \rightarrow 0$ , the amplitude  $\mathcal{M}_4^{\text{off-shell}}$  can only depend on the conformal cross-ratios  $u$  and  $v$  which, for the special choice  $p_i^2 = -m^2$ , are given by (20). Thus, the only conformally invariant variable is  $m^4/st$ , while the ratio  $s/t$  is not conformal. We are then lead to the conclusion that the term in (33), involving  $\ln^2(s/t)$  and originating from the form factors, must cancel against a similar term coming from  $\mathcal{F}_4^{\text{off-shell}}$ . Moreover, any further dependence on  $s/t$  in  $\mathcal{F}_4^{\text{off-shell}}$  is ruled out by the assumption of dual conformal invariance. In this way, we arrive at

$$\ln \mathcal{F}_4^{\text{off-shell}}\left(\frac{s}{t}\right) = \frac{\Gamma_{\text{cusp}}(a)}{4} \ln^2 \frac{s}{t} + \text{const} , \quad (34)$$

which is precisely the form (30) observed in [17] in perturbation theory and in [27] at strong coupling. We can interpret this as a manifestation of the dual conformal structure of four-gluon amplitudes.

## 4 Light-like Wilson loops

In this section we discuss the relation between the four-gluon scattering amplitude and the expectation value of a Wilson loop,

$$W_C = \frac{1}{N} \langle 0 | \text{Tr } P \exp \left( ig \int_C dx^\mu A_\mu(x) \right) | 0 \rangle. \quad (35)$$

Here  $A_\mu(x) = A_\mu^a(x) t^a$  is a gauge field and  $t^a$  are the  $SU(N)$  generators in the fundamental representation,  $C$  is a closed contour in Minkowski space and  $P$  stands for path ordering of the  $SU(N)$  indices. To the lowest order in the coupling it is given by

$$W_C = 1 + \frac{1}{2} (ig)^2 C_F \int_C dx^\mu \int_C dy^\nu G_{\mu\nu}(x - y) + O(g^4) \quad (36)$$

where  $C_F = (N^2 - 1)/(2N)$  is the Casimir of the fundamental representation of  $SU(N)$  and  $G_{\mu\nu}(x - y)$  is the free gluon propagator in a particular gauge. Since the Wilson loop (35) is a gauge invariant functional of the integration contour  $C$ , we can choose the gauge for convenience.

Following [27], we choose the integration contour in (35) to consist of four light-like segments joining the points  $x_i^\mu$  (with  $i = 1, 2, 3, 4$ ) such that  $x_i - x_{i+1} = p_i$  coincide with the external on-shell momenta of the four-gluon scattering amplitude (recall the relation (3)). We would like to stress that this Wilson loop has the following unusual feature – the integration contour  $C$  is defined by the external momenta of the four-gluon scattering amplitude, but the gluon propagator in (36) is the one in configuration space, not in momentum space. Similar Wilson loops have already been discussed in the past in the context of the infrared asymptotics of the QCD scattering amplitudes and their properties were studied in Refs. [5, 13, 45, 46].

Due to Lorentz invariance, the Wilson loop  $W_C$  is a function of the scalar products  $(x_i \cdot x_j)$ . The conformal symmetry of  $\mathcal{N} = 4$  SYM imposes additional constraints on the possible form of this function. If the Wilson loop  $W(C)$  were well defined in four dimensions, one would deduce that it changes under the  $SO(2, 4)$  conformal transformations (translations, rotations, dilatations and special conformal transformations) as  $W_C = W_{C'}$  where the contour  $C'$  is the image of  $C$  under these transformations. In this way, one would conclude that  $W(C)$  could only depend on the two conformal invariant cross-ratios  $u$  and  $v$  defined in (8). However, for the Wilson loop under consideration the conformal symmetry becomes anomalous due to the presence of cusps on the integration contour  $C$ . These cusps lead to UV divergences in  $W_C$  which need to be regularized.

### 4.1 Cusp anomaly

To illustrate the cusp anomaly, let us evaluate the double contour integral in (36). In what follows we shall employ dimensional regularization  $D = 4 - 2\epsilon_{\text{UV}}$  (with  $\epsilon_{\text{UV}} > 0$ ; notice the difference with the IR regulator  $\epsilon_{\text{IR}} < 0$ ) and use the notation of [45]. By virtue of gauge invariance, we can choose the Feynman gauge in which the gluon propagator has the form

$$G_{\mu\nu}(x) = -g_{\mu\nu} \frac{\Gamma(1 - \epsilon_{\text{UV}})}{4\pi^2} (-x^2 + i0)^{-1 + \epsilon_{\text{UV}}} (\pi \tilde{\mu}^2)^{\epsilon_{\text{UV}}} \quad (37)$$

and perform the calculation directly in configuration space. For a reason which will become clear in a moment, we introduced the notation  $\tilde{\mu}^2$  for dimensionful parameter to distinguish it from

the similar parameter  $\mu^2$  that we used to regularize IR divergences in Section 3. Splitting the integration contour into four segments  $C = C_1 \cup C_2 \cup C_3 \cup C_4$  and introducing the parametrization  $C_i = \{x^\mu(\tau_i) = x_i^\mu - \tau_i p_i^\mu | 0 \leq \tau_i \leq 1\}$ , we find from (36)

$$\ln W_C = -\frac{g^2 C_F}{4\pi^2} \sum_{1 \leq j \leq k \leq 4} I_{jk} + O(g^4) \quad (38)$$

where

$$I_{jk} = - \int_0^1 d\tau_j \int_0^1 d\tau_k \frac{(p_j \cdot p_k) \Gamma(1 - \epsilon_{UV}) (\pi \tilde{\mu}^2)^{\epsilon_{UV}}}{[-(x_j - x_k - \tau_j p_j + \tau_k p_k)^2 + i0]^{1-\epsilon_{UV}}} \quad (39)$$

with  $x_i - x_{i+1} = p_i$ . It is convenient to represent these integrals by the Feynman diagrams shown

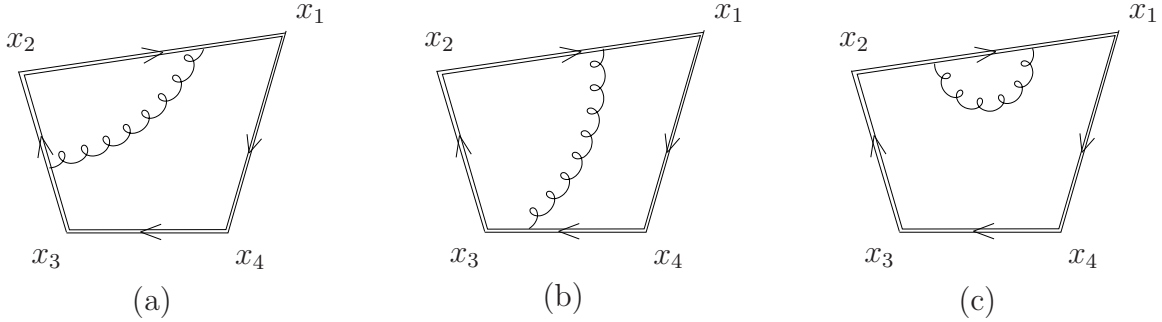


Figure 6: The Feynman diagram representation of the integrals (39): (a)  $I_{12}$ , (b)  $I_{13}$  and (c)  $I_{11}$ . The double line depicts the integration contour  $C$  and the wiggly line the gluon propagator.

in Fig. 6.

Then, to one-loop accuracy we encounter three types of integrals depicted in Fig. 6(a), (b) and (c). We begin with the last one and take into account the on-shell condition  $p_j^2 = 0$  to find  $I_{jj} \sim p_j^2 = 0$ . Let us now examine the integral  $I_{12}$  corresponding to Fig. 6(a)

$$I_{12} = - \int_0^1 d\tau_1 \int_0^1 d\tau_2 \frac{(p_1 \cdot p_2) \Gamma(1 - \epsilon_{UV}) (\pi \tilde{\mu}^2)^{\epsilon_{UV}}}{[-2(p_1 \cdot p_2)(1 - \tau_1)\tau_2]^{1-\epsilon_{UV}}}, \quad (40)$$

so that the integration over  $\tau_1$  and  $\tau_2$  yields

$$I_{12} = (\pi \tilde{\mu}^2(-s))^{\epsilon_{UV}} \frac{\Gamma(1 - \epsilon_{UV})}{2\epsilon_{UV}^2}. \quad (41)$$

Here the double pole in  $\epsilon_{UV}$  comes from integration in the vicinity of the cusp located at point  $x_2$  and has a clear ultraviolet origin.<sup>8</sup> It is easy to see from (39) that  $I_{34} = I_{12}$  whereas the integrals  $I_{23} = I_{14}$  can be obtained from  $I_{12}$  by substituting  $s \mapsto t$  with  $s = (p_1 + p_2)^2$  and  $t = (p_2 + p_3)^2$  being the usual Mandelstam variables in the momentum space of the four-gluon amplitude. The remaining integrals  $I_{13} = I_{24}$  are computed in a similar manner. We first verify that the integral  $I_{13}$  depicted in Fig. 6(b) remains finite for  $\epsilon_{UV} \rightarrow 0$  and, therefore, can be evaluated in  $D = 4$ . In this way, we get

$$I_{13} = \int_0^1 d\tau_1 \int_0^1 d\tau_3 \frac{(p_1 \cdot p_3)}{[p_1(1 - \tau_1) + p_2 + p_3\tau_3]^2} = -\frac{1}{2} \int_0^1 d\tau_1 \int_0^1 d\tau_3 \frac{s + t}{s\bar{\tau}_1 + t\tau_3 + st\bar{\tau}_1\tau_3} \quad (42)$$

<sup>8</sup>By ‘UV’ here we mean small distances in the dual ‘configuration’ space of the  $x_i$ .



with  $\bar{\tau}_1 = 1 - \tau_1$ . The integration can be easily performed and yields

$$I_{13} = -\frac{1}{4} [\ln^2(s/t) + \pi^2] + O(\epsilon_{\text{UV}}). \quad (43)$$

We can then replace the integrals  $I_{ik}$  in (38) by their expressions (41) and (43) and evaluate the one-loop correction to the light-like Wilson loop. Notice that in Eq. (41) the scale  $\tilde{\mu}^2$  has the ‘wrong’ dimension of  $[\text{mass}]^{-2}$  as compared to the conventional dimensional regularization scale. This is due to the fact that the contour  $C$  is defined in terms of the dual coordinates  $x_i$ , which in turn are related to the momenta  $p_i$  through (3). It is then convenient to introduce the new scale

$$\mu_{\text{UV}}^2 = (\tilde{\mu}^2 \pi e^\gamma)^{-1} \quad (44)$$

and to rewrite the one-loop expression for the Wilson loop (38) in the multi-color limit as follows

$$\ln W_C = a \left\{ -\frac{1}{\epsilon_{\text{UV}}^2} \left[ \left( \frac{\mu_{\text{UV}}^2}{-s} \right)^{-\epsilon_{\text{UV}}} + \left( \frac{\mu_{\text{UV}}^2}{-t} \right)^{-\epsilon_{\text{UV}}} \right] + \frac{1}{2} \ln^2 \frac{s}{t} + 2\zeta_2 + O(\epsilon_{\text{UV}}) \right\} + O(a^2). \quad (45)$$

Comparing this relation with the one-loop expression for the four-gluon scattering amplitude, Eqs. (16) – (18), we observe that, firstly, the divergent parts of the two expressions coincide provided that we formally identify the UV cutoff for the Wilson loop with the IR cutoff for the scattering amplitude,  $\mu_{\text{UV}}^2 = \mu_{\text{IR}}^2$ , and the IR regulator  $\epsilon_{\text{IR}}$  with the UV one  $\epsilon_{\text{UV}}$  (remembering, however, that  $\epsilon_{\text{IR}}$  and  $\epsilon_{\text{UV}}$  have different signs),  $\epsilon_{\text{IR}} = -\epsilon_{\text{UV}}$ . Secondly, the finite part of the Wilson loop contains the same  $\ln^2(s/t)$  term as the scattering amplitude (18), while the additive constants are different.

We can interpret this result as an indication that the duality between four-gluon scattering amplitudes and light-like Wilson loops proposed at strong coupling by Alday and Maldacena [27] also exists at weak coupling. However, the explicit form of the duality transformation which relates the two objects in  $\mathcal{N} = 4$  SYM remains to be found.

## 4.2 Evolution equations

A natural question to ask is whether the same duality between the four-gluon scattering amplitude and the light-like Wilson loop will survive at higher loops in the planar limit. For the divergent part of the two quantities the answer can be found by making use of the evolution equations. For the on-shell four-gluon scattering amplitude, one finds from (29) that its dependence on the IR scale  $\mu_{\text{IR}}$  is given by

$$\frac{\partial}{\partial \ln \mu_{\text{IR}}^2} \ln \mathcal{M}_4 = -\frac{1}{2} \Gamma_{\text{cusp}}(a) \ln \left( \frac{\mu_{\text{IR}}^4}{st} \right) - G(a) - \frac{1}{\epsilon_{\text{IR}}} \int_0^a \frac{da'}{a'} \Gamma_{\text{cusp}}(a') + O(\epsilon_{\text{IR}}), \quad (46)$$

where  $G(a) = \sum_l a^l G^{(l)}$  is the so-called collinear anomalous dimension. It has been calculated in [17] to three-loop accuracy:

$$G(a) = -\zeta_3 a^2 + \left( 4\zeta_5 + \frac{10}{3} \zeta_2 \zeta_3 \right) a^3 + O(a^4). \quad (47)$$

The three-loop expression for the cusp anomalous dimension reads [17]

$$\Gamma_{\text{cusp}}(a) = 2a - 2\zeta_2 a^2 + 11\zeta_4 a^3 + O(a^4). \quad (48)$$

For the light-like Wilson loop under consideration, a similar relation follows from its renormalization properties [45]:

$$\frac{\partial}{\partial \ln \mu_{\text{UV}}^2} \ln W_C = -\frac{1}{2} \Gamma_{\text{cusp}}(a) \ln \left( \frac{\mu_{\text{UV}}^4}{st} \right) - \Gamma(a) + \frac{1}{\epsilon_{\text{UV}}} \int_0^a \frac{da'}{a'} \Gamma_{\text{cusp}}(a') + O(\epsilon_{\text{UV}}) \quad (49)$$

with the anomalous dimension  $\Gamma(a) = 0 \cdot a + O(a^2)$ . The relation (49) can also be interpreted as the dilatation Ward identity for the Wilson loop in  $\mathcal{N} = 4$  SYM. Comparing relations (46) and (49), we conclude that the IR divergent part of  $\ln \mathcal{M}_4$  matches the UV divergent part of the dual light-like Wilson loop to all orders provided that the corresponding scales are related to each other as follows:

$$\ln \frac{\mu_{\text{UV}}^2}{\mu_{\text{IR}}^2} = \frac{G(a) - \Gamma(a)}{\Gamma_{\text{cusp}}(a)}. \quad (50)$$

Since the anomalous dimensions  $G(a)$  and  $\Gamma(a)$  receive perturbative corrections starting from two loops only, the right-hand side of this relation is given by a series in the coupling constant  $a$ .

To evaluate the normalization factor entering the scale-setting relation (50), one has to determine the Wilson loop anomalous dimension  $\Gamma(a)$  to two loops and supplement it with the known results for  $G(a)$  and  $\Gamma_{\text{cusp}}(a)$ . Since  $\Gamma(a)$  does not depend on the Mandelstam variables  $s, t$ , we may simplify the analysis by choosing  $t = -s$ , or equivalently  $p_2 = -p_3$  and  $p_1 = -p_4$ . It is easy to see that the resulting integration contour  $C$  takes the form of a rhombus with its parallel sides along two different light-cone directions. Such a Wilson loop has been studied in QCD in the context of the gluon Regge trajectory [32] (see the next section) and the same two-loop expression for  $W_C$  has been found in two different gauges [45] (Feynman and light-like axial gauge). It is straightforward to generalize the two-loop QCD result to  $\mathcal{N} = 4$  SYM [20]. To this end one has to add to  $W_C$  the contribution of  $n_s = 6N$  scalars,  $n_f = 4N$  gauginos (to two loops, they only enter through the one-loop correction to the gluon polarization operator) and convert the result from the dimensional regularization scheme (DREG) to the dimensional reduction scheme (DRED). In this way we have found that for  $s = -t$  the two-loop Wilson loop  $W_C$  satisfies the evolution equation (49) with the following value of the two-loop anomalous dimension

$$\Gamma(a) = -7\zeta_3 a^2 + O(a^3). \quad (51)$$

Substituting this relation into (50) and taking into account (47) and (48), we finally find

$$\ln \frac{\mu_{\text{UV}}^2}{\mu_{\text{IR}}^2} = 3\zeta_3 a + O(a^2). \quad (52)$$

Under such an identification of the scales, the light-like Wilson loop  $\ln W_C$  matches the on-shell four-gluon scattering amplitude  $\ln \mathcal{M}_4(s, t = -s)$  to two loops up to finite,  $s$ -independent constant terms.

Obviously, the calculation of the Wilson loop for  $s = -t$  does not allow us to test the form of the finite part. It would be interesting to carry out a full two-loop calculation of the Wilson loop to see if the duality gluon amplitudes/Wilson loops also applies to the functional dependence on  $s, t$ , as it did at one loop.

### 4.3 Conformal invariance of the dual Wilson loops

As was already mentioned, the cusp anomaly breaks the  $SO(2, 4)$  conformal symmetry of the four-dimensional Wilson loops. If this anomaly was not present, the Wilson loop would be a function

of the conformally invariant cross-ratios (8) only. Notice that the cusp anomaly originates from the integration in the vicinity of the cusps and, as a consequence, its contribution depends on the corresponding cusp angle or, equivalently, on the scalar products  $2(p_i \cdot p_{i+1})$ . In other words, each cusp produces an additive contribution to  $\ln W_C$  depending either on  $s$ , or on  $t$  but not on both variables simultaneously. This suggests that the ‘crossed’ terms  $\sim \ln(-s) \ln(-t)$  in the perturbative expansion of  $\ln W_C$  will not be affected by the cusp anomaly and, therefore, their form is still subject to the conformal symmetry constraints. Indeed, we already observed that the finite  $\ln^2(s/t)$  contribution to  $\ln W_C$  matches the similar contribution to  $\ln \mathcal{M}_4$ , which in turn follows from the conjectured off-shell conformal symmetry of the scattering amplitude. In the dual, Wilson loop description this symmetry is just the conformal symmetry of the four-dimensional Wilson loops.

Above we have demonstrated that the IR divergences of the on-shell scattering amplitude are dual to the UV (cusp) divergences of the light-like Wilson loop. One may wonder whether the same correspondence would survive if one assigned off-shellness (virtuality) to the momenta,  $p_i^2 = -m^2$ . In this case, the off-shell scattering amplitude is well defined in four dimensions while the cusp anomaly is still present in the Wilson loop. It produces a divergent contribution to  $W_C$  which satisfies the following evolution equation

$$\frac{\partial}{\partial \ln \mu_{\text{UV}}^2} \ln W_C^{\text{off-shell}} = -\frac{1}{2} \Gamma_{\text{cusp}}(a) \ln \left( \frac{m^4}{st} \right) + O(\epsilon_{\text{UV}}). \quad (53)$$

This implies that the (finite) off-shell scattering amplitude  $\mathcal{M}_4^{\text{off-shell}}$  cannot be identified with the (divergent) Wilson loop  $W_C^{\text{off-shell}}$  evaluated along the same contour  $C$  as before with the only difference being that  $p_i^2 = -m^2$ , or equivalently, that the segments run along space-like directions. This indicates that the precise nature of the duality between off-shell gluon amplitudes and Wilson loops needs to be investigated in more depth.

## 5 Gluon Regge trajectory

In this section we examine the asymptotic behaviour of the on-shell four-gluon scattering amplitude (29) in the Regge limit

$$s > 0, \quad t < 0, \quad s \gg -t. \quad (54)$$

It is expected that the scattering amplitude in this limit is given by the sum over Regge trajectories, each producing a power-like contribution  $\mathcal{M}_4(s, t) \sim s^{\omega(-t)}$ . In the planar limit, only the trajectory with the quantum numbers of a gluon gives a dominant contribution, so that the sum over the Regge trajectories is reduced to a single contribution from the gluon Regge trajectory [47, 48],

$$\mathcal{M}_4(s, t) = [c(-t)]^2 \left( \frac{s}{-t} \right)^{\omega_R(-t)} + [\text{subleading terms in } |t|/s], \quad (55)$$

where  $\omega_R(-t)$  is the Regge trajectory and  $c(-t)$  is the gluon impact factor. It is believed that the relation (55) holds in generic gauge theories ranging from QCD to  $\mathcal{N} = 4$  SYM. Indeed, the gluon reggeization was first discovered in QCD [48] and the gluon Regge trajectory is presently known to two-loop accuracy [49]. Moreover, it has been shown in [32] that the all-loop gluon Regge trajectory takes the following form in QCD

$$\omega_R^{(\text{QCD})}(-t) = \frac{1}{2} \int_{(-t)}^{\mu_{\text{IR}}^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \Gamma_{\text{cusp}}(a(k_{\perp}^2)) + \Gamma_R(a(-t)) + [\text{poles in } 1/\epsilon_{\text{IR}}], \quad (56)$$

where the two terms on the right-hand side define finite (as  $\epsilon_{\text{IR}} \rightarrow 0$ ) contributions. . Also, since the QCD beta-function is different from zero, the coupling constant depends on the normalization scale as indicated in (56). The anomalous dimension  $\Gamma_R(a)$  is given to two loops by

$$\Gamma_R(a) = 0 \cdot a + a^2 \left[ \frac{101}{27} - \frac{1}{2}\zeta_3 - \frac{14}{27} \frac{n_f}{N} \right] + O(a^3) \quad (57)$$

where  $n_f$  is the number of quark flavors.

We would like to stress that for generic values of the Mandelstam variables the four-gluon QCD scattering amplitude has a rather complicated form [50] different from (55). It is only in the Regge limit that one recovers (55) as describing the leading asymptotic behaviour of the four-gluon QCD scattering amplitude in the planar approximation [51]. One would expect that a similar simplification should also take place for the four-gluon planar amplitude in  $\mathcal{N} = 4$  SYM in the Regge limit. As we will see in a moment, this amplitude has the remarkable property of being *Regge exact*, i.e. the contribution of the gluon Regge trajectory to the amplitude (55) coincides with the exact expression for  $\mathcal{M}_4(s, t)$  with  $s$  and  $t$  arbitrary.

If the relation (55) is exact, i.e. if the subleading terms in the right-hand side of (55) are absent, then the scattering amplitude has to satisfy the evolution equation

$$\frac{\partial}{\partial \ln s} \ln \mathcal{M}_4(s, t) = \omega_R(-t) . \quad (58)$$

Notice that the right-hand side of this relation should be  $s$ -independent up to terms vanishing as  $\epsilon_{\text{IR}} \rightarrow 0$ . Let us verify this relation using the factorized expression (29) for  $\ln \mathcal{M}_4$ :

$$\omega_R(-t) = \frac{\partial}{\partial \ln s} \left[ \ln M^{gg \rightarrow 1} \left( \frac{\mu_{\text{IR}}^2}{-s}, \epsilon_{\text{IR}} \right) + \ln \mathcal{F}_{\text{on-shell}} \left( \frac{s}{t} \right) \right] . \quad (59)$$

Taking into account (28) and (30) we find that, in agreement with our expectations, the  $s$ -dependence disappears in the sum of the two terms and we obtain the following gluon trajectory:

$$\omega_R(-t) = \frac{1}{2} \Gamma_{\text{cusp}}(a) \ln \frac{\mu_{\text{IR}}^2}{(-t)} + \frac{1}{2} G(a) + \frac{1}{2\epsilon_{\text{IR}}} \int_0^a \frac{da'}{a'} \Gamma_{\text{cusp}}(a') + O(\epsilon_{\text{IR}}) . \quad (60)$$

Together with (47) and (48), this relation defines the gluon Regge trajectory in  $\mathcal{N} = 4$  SYM to three loops.

Let us compare the expressions for the gluon trajectory in QCD and in  $\mathcal{N} = 4$  SYM, Eqs. (56) and (60), respectively. It is easy to see that (60) can be obtained from (56) if we neglect the running of the coupling constant (recall that the beta function vanishes in  $\mathcal{N} = 4$  SYM to all loops) and identify the anomalous dimension  $\frac{1}{2}G(a)$  with  $\Gamma_R(a)$  in  $\mathcal{N} = 4$  SYM. Moreover, comparing the two-loop expression (57) for  $\Gamma_R(a)$  with  $\frac{1}{2}G(a) = -\frac{1}{2}\zeta_3 a^2 + O(a^3)$  (see (47)), we observe that the latter can be obtained from the former by retaining the terms of maximal transcendentality only.

The analysis performed in Section 4 suggests that in  $\mathcal{N} = 4$  SYM at weak coupling the four-gluon planar amplitude  $\ln \mathcal{M}_4(s, t)$  matches, for arbitrary  $s$  and  $t$ , the expectation value of the light-like Wilson loop  $\ln W_C$  up to an additive constant term. Then, going to the Regge limit (54) and making use of the relation (55), we can identify the gluon Regge trajectory in terms of the dual Wilson loop. The relation between these two seemingly different objects was first

observed in QCD in Ref. [32].<sup>9</sup> The Wilson loop (35) is defined in QCD in the same way as in  $\mathcal{N} = 4$  SYM. At weak coupling, the one-loop expressions for  $W_C$  are the same in the two theories but they differ from each other starting from two loops. It was observed in [32] that the two-loop QCD expression for the Wilson loop  $W_C$ , with  $C$  being a light-like rectangular loop, takes the Regge-like form (55). Moreover, the resulting two-loop expression for the exponent of  $s$  can be brought to the same form as the two-loop gluon Regge trajectory (56). This is achieved by replacing the IR cutoff  $\mu_{\text{IR}}^2$  by the UV cutoff  $\mu_{\text{UV}}^2$ , and the two-loop anomalous dimension  $\Gamma_R(a)$  in the right-hand side of (56) by

$$\Gamma_R^{(\text{dual})}(a) = 0 \cdot a + a^2 \left[ \frac{101}{27} - \frac{7}{2}\zeta_3 - \frac{14}{27} \frac{n_f}{N_c} \right] + \mathcal{O}(a^3). \quad (61)$$

Then we express  $\mu_{\text{UV}}^2$  in terms of  $\mu_{\text{IR}}^2$  with the help of (50) and take into account the relation  $\Gamma_R(a) - \Gamma_R^{(\text{dual})}(a) = 3\zeta_3 a^2 + \mathcal{O}(a^3)$  to verify that the Regge trajectory obtained from the Wilson loop calculation coincides with the two-loop expression for the gluon Regge trajectory in QCD (56).

Needless to say, QCD is very different from  $\mathcal{N} = 4$  SYM and its dual string description is not known yet. Nevertheless, the very fact that the two-loop gluon Regge trajectory in QCD admits a dual description in terms of a light-like Wilson loop provides yet another indication that QCD possesses some hidden (integrable) structure [52].

## 6 Summary and discussion

In this paper we have presented further evidence for a dual conformal symmetry in the planar four-gluon amplitude in  $\mathcal{N} = 4$  SYM. We have shown that all the momentum loop integrals appearing in the perturbative calculations up to five loops are dual to true conformal integrals, well defined off shell. Assuming that the complete off-shell amplitude has this property, we have derived the special form of the finite remainder previously found perturbatively and reproduced at strong coupling by AdS/CFT. We have also shown that the same finite term appears in a weak coupling calculation of a Wilson loop whose contour consists of four light-like segments associated with the gluon momenta. We have demonstrated that the dual conformal symmetry leads to dramatic simplification of the planar four-gluon amplitude in the high-energy (Regge) limit. Namely, due to the special form of the finite remainder, the contribution of the gluon Regge trajectory to the amplitude coincides with its exact expression evaluated for arbitrary values of the Mandelstam variables.

Several questions remain open. First of all, we need to investigate the four-gluon amplitude in the off-shell regime beyond one loop. One should not take it for granted that going off shell simply consists in changing the regulator in the set of loop integrals contributing to the on-shell amplitude. It is quite likely that new integrals will appear off shell, such that they vanish for  $p_i^2 = 0$  in dimensional regularization but contribute when  $p_i^2 \rightarrow 0$  in  $D = 4$ . There are good reasons to believe that this will indeed be the case [53, 54]. The crucial question will then be whether the dual conformal symmetry concerns these additional contributions as well. A two-loop calculation of the off-shell amplitude, which may help clarify this point, is currently under way.

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<sup>9</sup>We would like to stress that, in distinction with  $\mathcal{N} = 4$  SYM, the four-gluon planar amplitude in QCD is not dual to light-like Wilson loop for arbitrary  $s$  and  $t$ . The relation between the two quantities emerges in the Regge limit only.

Another question is if the dual conformal symmetry is specific to four-gluon amplitudes or also applies to multi-gluon ones. The iteration conjecture of [17] predicts the exponentiation of the one-loop finite part independently of the number of external legs. This has so far been confirmed by an explicit two-loop five-gluon calculation in [25, 26]. One has to analyze the integrals appearing there to see if they possess similar conformal properties. It should be pointed out that the number of conformally invariant variables (cross-ratios) rapidly grows with the number of points, therefore multiple-point dual conformal symmetry, if present, may be less restrictive than at four points.

A natural question to ask is whether going off shell does not break gauge invariance, thus possibly spoiling the on-shell dual conformal symmetry. We can answer this question in the following way. In the off-shell regime we expect two phenomena to take place. Firstly, the double-log singularities get additional contributions from a new subprocess associated with exchanges of particles carrying infrared (or ‘ultra-soft’) momenta [38, 41]. This contribution is cancelled, however, in the finite part of the four-gluon amplitude defined as the ratio of the off-shell scattering amplitude and the off-shell form factors. At the same time, the finite part of the off-shell amplitude receives  $s/t$ -dependent contributions from a hard subprocess in which the particle momenta are of order  $\sqrt{-s}$ ,  $\sqrt{-t}$ . The latter is not sensitive to the virtuality of the external legs (recall the one-loop example of Section 3.1). We thus expect that the gauge dependence of the finite part of the scattering amplitude may affect only the constant  $s/t$ -independent term which is scheme dependent anyway [40].

Finally, let us comment on the Wilson loop. In  $\mathcal{N} = 4$  SYM, the Wilson loop evaluated along a smooth closed contour is UV finite and has conformal symmetry. However, the presence of cusps causes specific ‘cusp’ UV divergences. To regularize them in the perturbative calculation, we had to introduce a dimensional regulator which breaks the conformal symmetry. Surprisingly enough, the finite part of the light-like Wilson loop with four cusps on the integration contour has exactly the same special form as the four-gluon amplitude. We are tempted to interpret this fact as a signal that the basic reason why the finite part is always the same is conformal symmetry. It appears to be broken in a controlled way in all three calculations (the perturbative on-shell gluon amplitude of Bern *et al*, its strong coupling dual of Alday and Maldacena, and our perturbative Wilson loop), leaving as a common trace the specific form of the finite part.

## Acknowledgments

We have profited from enlightening discussions with L. Alday, I. Antoniadis, Z. Bern, A. Davydchev, S. Ferrara, J. Henn, D. Kosower, L. Lipatov, J. Maldacena, V. Smirnov and R. Stora. Special acknowledgements are due to A. Brandhuber, P. Heslop and G. Travaglini and to L. Dixon who have shared with us their observations about the exponentiation properties of the on-shell four-gluon amplitude. ES is very grateful to the Department of Physics and Astronomy at UCLA and especially to Zvi Bern for hospitality during the initial stages of this work. This research has been partially supported by the French Agence Nationale de la Recherche, contract ANR-06-BLAN-0142.



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